CONTINUANT EXPRESSIONS FOR $\sqrt{a^2+b}$ AND $(\sqrt{a^2+b}+a)^n$.

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The line of reasoning employed in a paper by Muir* is applicable to a more general expression than that with which his paper was concerned.

We shall first show that *if the positive integral powers of* $\sqrt{a^2+b}+a$ *be taken, and the expansion of each be separated into two parts, rational and irrational, then the ratio of the rational portion to the coefficient of* $\sqrt{a^2+b}$ *approaches as a limit* $\sqrt{a^2+b}$ *or* $-\sqrt{a^2+b}$, *as the index of the power approaches infinity, according as a is positive or negative.*

Manifestly,

$$(\sqrt{a^2+b}+a)^n = \frac{(\sqrt{a^2+b}+a)^n + (-1)^{n-1}(\sqrt{a^2+b}-a)^n}{2} + \frac{(\sqrt{a^2+b}+a)^n + (-1)^n(\sqrt{a^2+b}-a)^n}{2};$$

and in this expression the first fraction always contains $\sqrt{a^2+b}$ as a factor, while the second fraction is always rational. Consequently we write

$$(\sqrt{a^2+b}+a)^n = \frac{(\sqrt{a^2+b}+a)^n + (-1)^{n-1}(\sqrt{a^2+b}-a)^n}{2\sqrt{a^2+b}}\sqrt{a^2+b} + \frac{(\sqrt{a^2+b}+a)^n + (-1)^n(\sqrt{a^2+b}-a)^n}{2}, \tag{1}$$

thereby separating the expansion as specified. The nth convergent,

$$\frac{(\sqrt{a^2+b}+a)^n + (-1)^n(\sqrt{a^2+b}-a)^n}{(\sqrt{a^2+b}+a)^n + (-1)^{n-1}(\sqrt{a^2+b}-a)^n}\sqrt{a^2+b},$$

may be put into either of the forms

* Muir, Thos., "Note on a theorem regarding a series of convergents to the roots of a number," Proc. Roy. Soc. Edin., vol. XIX, p. 15.

$$\frac{1+(-1)^n\left(\dfrac{\sqrt{a^2+b}-a}{\sqrt{a^2+b}+a}\right)^n}{1+(-1)^{n-1}\left(\dfrac{\sqrt{a^2+b}-a}{\sqrt{a^2+b}+a}\right)^n}\sqrt{a^2+b}, \qquad \frac{\left(\dfrac{\sqrt{a^2+b}+a}{\sqrt{a^2+b}-a}\right)^n+(-1)^n}{\left(\dfrac{\sqrt{a^2+b}+a}{\sqrt{a^2+b}-a}\right)^n+(-1)^{n-1}}\sqrt{a^2+b}.$$

If a is positive, the first form shows that the limit is $\sqrt{a^2+b}$; if a is negative, the second form shows that the limit is $-\sqrt{a^2+b}$.

Ramus,* in 1856, obtained a result which we may express in the form

$$\begin{vmatrix} a & b & & \\ -1 & a & b & \\ & -1 & a & b \\ & & \cdot\;\cdot & \cdot \end{vmatrix}_{n-1} = \frac{1}{\sqrt{a^2+4b}}\left[\left(\frac{a+\sqrt{a^2+4b}}{2}\right)^n-\left(\frac{a-\sqrt{a^2+4b}}{2}\right)^n\right],$$

or, replacing a by $2a$, and making a further obvious modification,

$$\begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \cdot\;\cdot & \cdot \end{vmatrix}_{n-1} = \frac{(\sqrt{a^2+b}+a)^n+(-1)^{n-1}(\sqrt{a^2+b}-a)^n}{2\sqrt{a^2+b}}.$$

We also have, from the properties of continuants,

$$\begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \cdot\;\cdot & \cdot \end{vmatrix}_{n} = a\begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & & \cdot\;\cdot & \cdot \end{vmatrix}_{n-1} + b\begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & & \cdot\;\cdot & \cdot \end{vmatrix}_{n-2},$$

whence, since $b=(\sqrt{a^2+b}+a)(\sqrt{a^2+b}-a)$, the continuant on the left of the last equation is equal to

$$\frac{\begin{array}{r} a(\sqrt{a^2+b}+a)^n+(-1)^{n-1}a(\sqrt{a^2+b}-a)^n+(\sqrt{a^2+b}-a)(\sqrt{a^2+b}+a)^n \\ +(-1)^{n-2}(\sqrt{a^2+b}+a)(\sqrt{a^2+b}-a)^n \end{array}}{2\sqrt{a^2+b}}$$

$$=\frac{(\sqrt{a^2+b}+a)^n+(-1)^n(\sqrt{a^2+b}-a)^n}{2},$$

* Ramus, C., "Determinanternes Anvendelse til at bestemme hoven for de convergerende Bröker." Oversigt . . . danske Vidensk. Selsk. Forhandl. . . . Kjøbenhavn, pp. 106–119. Muir's Theory of Determinants, vol. II, p. 427.

which is the rational term in (1). We may therefore rewrite (1) in the forms

$$(\sqrt{a^2+b}+a)^n = \begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & \cdot & \cdot & \cdot \end{vmatrix}_{n-1} \sqrt{a^2+b} + \begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \cdot\ \cdot & \cdot \end{vmatrix}_{n}$$

$$= \begin{vmatrix} 1 & \sqrt{a^2+b} & & & \\ -1 & a & b & & \\ & -1 & 2a & b & \\ & & -1 & 2a & b \ \cdot \\ & & & \cdot\ \cdot & \cdot \end{vmatrix}_{n+1} . \qquad (A)$$

We found that

$$\sqrt{a^2+b} = \begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \cdot\ \cdot & \cdot \end{vmatrix} \div \begin{vmatrix} 2a & b & \\ -1 & 2a & b \\ & \cdot\ \cdot & \cdot \end{vmatrix} \qquad (a > 0),$$

and

$$\sqrt{a^2+b} = -\begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \cdot\ \cdot & \cdot \end{vmatrix} \div \begin{vmatrix} 2a & b & \\ -1 & 2a & b \\ & \cdot\ \cdot & \cdot \end{vmatrix} \qquad (a < 0).$$

By the rule for changing the signs of the principal diagonal elements of a continuant, the latter equation becomes

$$\sqrt{a^2+b} = \begin{vmatrix} |a| & b & & \\ -1 & 2|a| & b & \\ & -1 & 2|a| & b \\ & & \cdot\ \cdot & \cdot \end{vmatrix} \div \begin{vmatrix} 2|a| & b & \\ -1 & 2|a| & b \\ & \cdot\ \cdot & \cdot \end{vmatrix},$$

which holds for both positive and negative values of a. Hence we have, finally,

$$\sqrt{a^2+b} = |a| + \frac{b}{2|a|+}\,\frac{b}{2|a|+}\cdots. \qquad (B)$$

A part of this result will be seen to furnish a proof of the truth of an equation put forth as a problem in Chrystal's Algebra, Part II, Exs. XXXI, No. (9).

In the proof leading up to equation (B) it is a necessary condition, in case b is negative, that $a^2 > |b|$.

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